

New Branches of String Compactifications and their F-Theory Duals

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Abstract

We study heterotic $E_8 \times E_8$ models that are dual to compactifications of F-theory and type IIA string on certain classes of elliptically fibered Calabi-Yau manifolds. Different choices for the specific torus in the fibration have heterotic duals that are most easily understood in terms of $E_8 \times E_8$ models with gauge backgrounds of type $H \times U(1)^{8-d}$, where H is a non-Abelian factor. The case with $d = 8$ corresponds to the well known $E_8 \times E_8$ compactifications with non-Abelian instanton backgrounds (k_1, k_2) whose F-theory duals are built through compactifications on fibrations of the torus $\mathbb{P}_2^{(1,2,3)}[6]$ over \mathbb{F}_n . The new cases with $d < 8$ correspond to other choices for the elliptic fiber over the same base and yield unbroken $U(1)$'s, some of which are anomalous and acquire a mass by swallowing zero modes of the antisymmetric B_{MN} field. We also study transitions to models with no tensor multiplets in $D = 6$ and find evidence of E_d instanton dynamics. We also consider the possibility of conifold transitions among spaces with different realization of the elliptic fiber.

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1 Introduction

The existence of strong-weak coupling dualities [1] in string theory seems to be firmly established by now. Many different strings on different vacua which were previously thought to be independent turn out to be connected in some manner by different string dualities. More specifically, the evidence supporting the idea [2, 3] of a strong-weak coupling duality between type IIA and heterotic strings has increased with the new insights provided from the perspective of F-theory [4, 5, 6]. In this article we wish to explore new branches of $K3$ and $K3 \times T^2$ heterotic compactifications and explain how they are related to F-theory and type IIA compactifications.

Our basic motivation is the observation of [7] that in many cases type II candidates to heterotic duals appear to be organized into chains, corresponding to sequential Higgsing in the heterotic side, following a very precise pattern. Duality requires the occurrence of transitions among the Calabi-Yau (CY) spaces in the type II side as well as enhancing of gauge symmetries due to the singularity structure of the manifold. For a particular chain of CY spaces ending in $\mathbb{P}_4^{(1,1,n,2n+4,3n+6)}[6n+12]$ and henceforth labeled as type A, subsequent work has confirmed the expected behavior thereby lending strong support to the duality conjecture [8, 9, 5, 10, 11, 12].

Based on the pattern of weight regularities, besides the A class, different classes B, C, ... of dual type II chains were postulated in [7] even though the heterotic models were not found at the time. Here we will show explicitly how to construct the required B, C, ... heterotic chains. We will see that these models have an intrinsic six-dimensional description in terms of $K3$ compactifications with non-semisimple $E_8 \times E_8$ backgrounds. In fact, A, B, C, ... models can be built up by embedding none, one, two, ... $U(1)$ backgrounds in each E_8 factor. An interesting feature of the new models is the presence of anomalous $U(1)$'s, that acquire mass at tree-level by swallowing zero modes of the antisymmetric B_{MN} field, together with non-anomalous $U(1)$'s whose breaking corresponds to transitions $\cdots C \rightarrow B \rightarrow A$.

F-theory has proved to be very fruitful for a geometric understanding of different string dualities [4, 5, 6]. In particular it was argued in [5] that F-theory compactifications to six dimensions on certain elliptically fibered CY 3-folds are dual to certain heterotic compactifications on $K3$. Upon further toroidal compactification on T^2 , type II/heterotic duality is naturally recovered. By extending the analysis of [5] we will be able to construct explicit F-theory duals for the new heterotic models. More-

over, each class of models will be shown to be associated to fibrations of different elliptic fibers over the base \mathbb{F}_n , thus establishing a correspondence between elliptic fiber on the F-theory side and $U(1)$ factors on the heterotic side. More precisely, A, B, C, \dots models correspond to elliptic fibrations where the elliptic fiber is respectively, $\mathbb{P}_2^{(1,2,3)}[6]$, $\mathbb{P}_2^{(1,1,2)}[4]$, $\mathbb{P}_2^{(1,1,1)}[3]$, \dots . We will also argue that from the point of view of type IIA compactifications the change of elliptic fiber appears to correspond to conifold transitions as suggested in [5].

This article is structured as follows. In Chapter 2 we introduce some basic concepts and notation and review the properties of the chains of models proposed in [7]. In Chapter 3 we explore heterotic $K3$ and $K3 \times T^2$ compactifications with generic background embeddings in $E_8 \times E_8$ containing both Abelian and non-Abelian factors. A detailed case by case analysis of different assignments of instanton numbers indicates perfect agreement of the resulting spectra with the Hodge numbers of the CY chains of various types. New model building possibilities using semisimple non-Abelian backgrounds are also discussed and intrinsically four-dimensional heterotic chains involving enhancing of the toroidal $U(1)$'s are considered to some extent. In Chapter 4, F-theory duals are constructed. Different fibrations are studied and a singularity analysis is performed to identify enhanced gauge symmetries. Conifold transitions among different chains are also discussed. Chapter 5 is devoted to the study of transitions to models without tensor multiplets. Conclusions, miscellaneous results and general outlook are presented in Chapter 6.

2 Heterotic/Type II Duality and $D = 6$ Heterotic Dynamics

Type IIA compactifications are characterized by the CY Hodge numbers (b_{21}, b_{11}) where $b_{21} + 1$ counts the number of hypermultiplets (including the dilaton) and b_{11} counts the number of vector multiplets. The perturbative gauge group including the graviphoton is $U(1)^{b_{11}+1}$. On the other hand, $N = 2$ heterotic strings in general have gauge symmetry group G of $\text{rank}(G) = n_V + 1$ including the graviphoton. Here n_V counts the number of vector multiplets including the dilaton. Giving vevs to adjoint scalars in vector multiplets realizes the transition to the Coulomb phase in which the gauge group is generically broken to $U(1)^{n_V+1}$. A necessary requirement for

duality is therefore $(b_{21}, b_{11}) = (n_H - 1, n_V)$, where n_H is the number of (neutral) hypermultiplets that remain massless in the Coulomb phase. It is also required that the candidate CY dual be a $K3$ fibration [13, 14, 15].

In ref. [7] different models were constructed mainly by considering T^4/Z_M ($M = 2, 3, 4, 6$) orbifold limits of $K3$ and by embedding the orbifold action as a shift in the $E_8 \times E_8$ or $Spin(32)/Z_2$ gauge lattice. After compactification on T^2 , $N = 2$, $D = 4$ models were obtained. The rank of the starting gauge group was then reduced in steps by giving vevs to scalar in hypermultiplets. Moving to the Coulomb phase at each step, $(n_H - 1, n_V)$ was compared with candidate CY Hodge numbers. This produced the chains of models of table 1 in [7]. We will refer to these as chains of A type.

A unified and extended version of these A models can be obtained by considering $K3$ compactifications with instanton backgrounds in $E_8 \times E_8$. Let us denote by $H_{1,2}$ the background gauge (simple) groups and by (k_1, k_2) the corresponding instanton numbers. From each E_8 the unbroken gauge group is the commutant G_i of H_i . The adjoint representation of E_8 decomposes under $G \times H$ as $\mathbf{248} = \sum_a (R_a, M_a)$. The number of hypermultiplets in the representation R_a of the unbroken group G is then computed from the index theorem

$$N(R_a) = k T(M_a) - \dim M_a \quad (2.1)$$

where $T(M_a)$ is given by $\text{tr}(T_a^i T_a^j) = T(M_a) \delta_{ij}$, T_a^i being an H generator in the representation M_a ¹. For example, an $SU(2)$ bundle with instanton number k gives $(k - 4)/2$ multiplets in the $\mathbf{56}$ of E_7 and $(2k - 3)$ singlets

Since anomaly cancellation requires $k_1 + k_2 = 24$, it is convenient to define

$$k_1 = 12 + n \quad ; \quad k_2 = 12 - n \quad (2.2)$$

and, without loss of generality, assume $n \geq 0$. For $n \leq 8$ it is possible to have an $E_7 \times E_7$ unbroken gauge group with hypermultiplet content

$$\frac{1}{2}(8 + n)(\mathbf{56}, \mathbf{1}) + \frac{1}{2}(8 - n)(\mathbf{1}, \mathbf{56}) + 62(\mathbf{1}, \mathbf{1}) \quad (2.3)$$

Due to the pseudoreal character of the $\mathbf{56}$ of E_7 , odd values of n can also be considered. For $9 \leq n \leq 12$, k_2 is not large enough to support an $SU(2)$ background. The

¹ Our normalization is such that $T(fund) = \frac{1}{2}, 1, 3, 6$ and 30 for $SU(N)$, $SO(N)$, E_6 , E_7 and E_8 respectively.

instantons become small and turn into $12 - n$ extra tensor multiplets. The unbroken gauge group is now $E_7 \times E_8$ with matter content

$$\frac{1}{2}(8+n)(\mathbf{56}, \mathbf{1}) + (53+n)(\mathbf{1}, \mathbf{1}) \quad (2.4)$$

Models with various groups can be obtained from (2.3) and (2.4) by symmetry breaking.

Notice that the group from the second E_8 does not posses, in general, enough charged matter to be completely broken. Higgsing stops at some terminal group, depending on the value of n , with minimal or no charged matter. For instance E_8 , E_7 , E_6 , $SO(8)$ terminal groups are obtained for $n = 12, 8, 6, 4$ while complete breaking proceeds for $n = 2, 0$. On the other hand, the first E_7 can be completely Higgsed away. The type A chains in [7] are reproduced by cascade breaking through

$$\cdots \rightarrow SU(4) \rightarrow SU(3) \rightarrow SU(2) \rightarrow \emptyset \quad (2.5)$$

In these chains, the weights of the candidate dual CY hypersurfaces in projective space follow a well defined pattern of regularities. Namely, the cascade Higgsing (2.5) maps into the following sequence in the type II side

$$\begin{aligned} \mathbb{P}_5^{(1,1,w_1,w_2,w_3,w_4)} &\rightarrow \mathbb{P}_4^{(1,1,w_1,w_2,w_3)} \rightarrow \\ \mathbb{P}_4^{(1,1,w_1,w_2,w_3+w_1)} &\rightarrow \mathbb{P}_4^{(1,1,w_1,w_2+w_1,w_3+2w_1)} \end{aligned} \quad (2.6)$$

where for the \mathbb{P}_4 's the degree of the hypersurface defining equation is of course the sum of the weights and for the \mathbb{P}_5 there are two equations of appropriate degrees. Moreover, these transitions can be recast in terms of n . In fact, in A models the last steps of cascade Higgsing (2.5) have candidate CY duals

$$\begin{aligned} \mathbb{P}_5^{(1,1,n,n+4,n+6,n+8)}[2n+12, 2n+8] &\rightarrow \mathbb{P}_4^{(1,1,n,n+4,n+6)}[3n+12] \rightarrow \\ \mathbb{P}_4^{(1,1,n,n+4,2n+6)}[4n+12] &\rightarrow \mathbb{P}_4^{(1,1,n,2n+4,3n+6)}[6n+12] \end{aligned} \quad (2.7)$$

This structure also holds for odd values of n [10].

Encouraged by these regularities, new chains were proposed in [7] by reorganizing $K3$ fibrations in the list of [13] according to eq. (2.6), even if heterotic models were not known at that time. For example, there exist chains of $K3$ fibrations, in which b_{11} jumps by one in each step, that end at the penultimate stage in (2.6) and that for generic n have the form

$$\begin{aligned} \mathbb{P}_5^{(1,1,n,n+2,n+4,n+6)}[2n+8, 2n+6] &\rightarrow \mathbb{P}_4^{(1,1,n,n+2,n+4)}[3n+8] \rightarrow \\ \mathbb{P}_4^{(1,1,n,n+2,2n+4)}[4n+8] & \end{aligned} \quad (2.8)$$

We will refer to these as chains of models B. There are also chains of models C that have two elements given by

$$\mathbb{P}_5^{(1,1,n,n+2,n+2,n+4)}[2n+6, 2n+4] \rightarrow \mathbb{P}_4^{(1,1,n,n+2,n+2)}[3n+6] \quad (2.9)$$

Finally, there are chains of type D with single element

$$\mathbb{P}_5^{(1,1,n,n+2,n+2,n+2)}[2n+4, 2n+4] \quad (2.10)$$

The structure of the CY chains is summarized in Table 1. In each case n is restricted by the condition that the set of weights lead to a well defined CY space. For type A, $n \leq 12$ in agreement with the heterotic construction. For types B and C, the weights correspond to reflexive polyhedra only for $n \leq 8$ and $n \leq 6$ respectively. For models D, $n \leq 4$ is expected.

r	A $2 \leq n \leq 12$	B $2 \leq n \leq 8$
4	$\mathbb{P}_5^{(1,1,n,n+4,n+6,n+8)}[2n+12, 2n+8]$	$\mathbb{P}_5^{(1,1,n,n+2,n+4,n+6)}[2n+8, 2n+6]$ $\mathbb{P}_4^{(1,1,n,n+2,n+4)}[3n+8]$ $\mathbb{P}_4^{(1,1,n,n+2,2n+4)}[4n+8]$
3	$\mathbb{P}_4^{(1,1,n,n+4,n+6)}[3n+12]$	
2	$\mathbb{P}_4^{(1,1,n,n+4,2n+6)}[4n+12]$	
1	$\mathbb{P}_4^{(1,1,n,2n+4,3n+6)}[6n+12]$	
r	C $2 \leq n \leq 6$	D $2 \leq n \leq 4$
2	$\mathbb{P}_5^{(1,1,n,n+2,n+2,n+4)}[2n+6, 2n+4]$	$\mathbb{P}_5^{(1,1,n,n+2,n+2,n+2)}[2n+4, 2n+4]$
1	$\mathbb{P}_4^{(1,1,n,n+2,n+2)}[3n+6]$	

Table 1: Structure of the A,B,C and D chains.

The Hodge numbers for the terminal elements of each chain are given in Table 2 for future reference ². The expressions in Table 1 clearly do not apply to $n = 0$ nor to $n = 1$, since it is known, for instance, that $\mathbb{P}_4^{(1,1,1,6,9)}[18]$ is not a $K3$ fibration. However, these two values are naturally considered once we notice that the terminal spaces correspond to elliptic fibrations over \mathbb{F}_n that can be extended to $n = 0, 1$

²Most of these results, as well as those in eq. (2.11) below, appear in refs. [16, 13]. The remaining cases in \mathbb{P}_4 have been computed using the program POLYHEDRON written by P. Candelas. The numbers for the spaces in \mathbb{P}_5 were calculated by A. Klemm.

using the formalism of ref. [5]. In section 4 we will explain in more detail the elliptic fibration structure of the various models. It is also worth noticing that the Hodge numbers (b_{12}^r, b_{11}^r) for the chains in Table 1 can all be written in terms of the (b_{12}^1, b_{11}^1) recorded in Table 2. Specifically,

$$\begin{aligned}
\text{A : } \quad & b_{12}^2 = b_{12}^1 - (12n + 29) \\
& b_{12}^3 = b_{12}^1 - (18n + 46) \\
& b_{12}^4 = b_{12}^1 - (22n + 61) \\
\text{B : } \quad & b_{12}^2 = b_{12}^1 - (6n + 15) \\
& b_{12}^3 = b_{12}^1 - (10n + 26) \\
\text{C : } \quad & b_{12}^2 = b_{12}^1 - (4n + 11)
\end{aligned} \tag{2.11}$$

In all cases $b_{11}^r = b_{11}^1 + r - 1$. These results are tabulated in Table 3, at the end of the article, for the reader's convenience.

In section 3 we will develop the heterotic construction that reproduces systematically the terminal elements of type B, C and D. Moreover, we will show how un-Higgsing of $SU(r)$ factors leads to spectra that match the Hodge numbers of the chains given in (2.11). In the heterotic construction many more symmetry breaking patterns are possible. We then expect that the terminal CY spaces are continuously connected to points with generic enhanced gauge symmetries as shown recently for the A models [10, 12].

It is not clear from the preceding discussion if there exists any correspondence among models with same value of n . However, the results in Table 2 suggest that this is indeed the case. For instance, the $n = 4$, $(271, 7)$ model in chain A with $SO(8)$ terminal group corresponds to the $n = 4$ models $(164, 8)$ in chain B, $(111, 9)$ in chain C and $(76, 10)$ in chain D. We observe that the rank increases in one unit when $A \rightarrow B \rightarrow C \rightarrow D$. On the other hand, the number n_H of hypermultiplets decreases in each step. This can be taken as an indication of the presence of an extra $U(1)$ group for models B so that their unbroken gauge group would be $SO(8) \times U(1)$. Likewise, there would be two and three extra $U(1)$ factors for models C and D. The existence of charged matter with respect to these $U(1)$ groups would explain the decreasing in n_H . Similar arguments apply to other n 's. For values such as $n = 5$ our heterotic construction will also explain the horizontal behavior of b_{11} .

In the type A heterotic models the gauge group structure before going to the

n	A	B	C	D
0	(243,3)	(148,4)	(101,5)	(70,6)
1	(243,3)	(148,4)	(101,5)	(70,6)
2	(243,3)	(148,4)	(101,5)	(70,6)
3	(251,5)	(152,6)	(103,7)	(70,10)
4	(271,7)	(164,8)	(111,9)	(76,10)
5	(295,7)	(178,10)	(120,12)	
6	(321,9)	(194,10)	(131,11)	
7	(348,10)	(210,12)		
8	(376,10)	(227,11)		
9	(404,14)			
10	(433,13)			
11	(462,12)			
12	(491,11)			

Table 2: Hodge numbers (b_{21}^1, b_{11}^1) for the terminal spaces .

Coulomb branch is of the form $G \times U(1)^4$ where $U(1)^4$ arises in the toroidal compactification from six to four dimensions. Since T^2 is untouched we can interpret the A models as intrinsically corresponding to $N=1$ compactifications on $K3$. We will see that this is also the case for models B,C and D. It is then useful to recall some properties of $N = 1$ six-dimensional theories that are in a sense more constrained since being chiral they could have potential anomalies. In particular, the anomaly 8-form should factorize as

$$I_8 = (\text{tr } R^2 - v_\alpha \text{tr } F_\alpha^2)(\text{tr } R^2 - \tilde{v}_\alpha \text{tr } F_\alpha^2) \quad (2.12)$$

where α runs over the gauge factors. The coefficients v_α are fixed for each gauge group. They are given by $v_\alpha = 2, 1, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}, \frac{1}{30}$ for $SU(N), SO(N), F_4, E_6, E_7, E_8$ [17] for Kac-Moody level one. On the other hand, the \tilde{v}_α coefficients depend on the hypermultiplet content of each group. For instance $\tilde{v}_{E_7} = \frac{1}{6}(n_{56} - 4)$. Results for other groups can be found in ref. [17]. For generic gauge group $G = G_1 \times G_2$ with

G_1 and G_2 subgroups of the first and second E_8 obtained from backgrounds with instanton numbers $(12 + n, 12 - n)$, the following identity is satisfied

$$\frac{\tilde{v}_1}{v_1} = \frac{n}{2} \quad ; \quad \frac{\tilde{v}_2}{v_2} = -\frac{n}{2} \quad (2.13)$$

These relations remain valid at each step of possible Higgsing.

From the anomaly polynomial it follows that the gauge kinetic terms are proportional to [18]

$$-v_1(e^{-\phi} + \frac{n}{2}e^{\phi})\text{tr } F_1^2 - v_2(e^{-\phi} - \frac{n}{2}e^{\phi})\text{tr } F_2^2 \quad (2.14)$$

where F_i is the field strength of the unbroken groups G_i and ϕ is the scalar dilaton living in a 6d tensor multiplet. Heterotic/heterotic duality [19] is obtained for $n = 0$ if small instanton effects are taken into account [20]. It is also present in the $n = 2$ case [21] if one Higgses away the second group factor. In fact both cases $n = 0$ and $n = 2$ turn out to be connected if examined from the F theory point of view [5]. The coefficient of the gauge kinetic term for the second E_8 is such that the gauge coupling diverges at [20, 22]

$$e^{-2\phi} = \frac{n}{2} \quad (2.15)$$

This is a sign of a phase transition in which there appear tensionless strings [23, 22, 24].

In the previous discussion of six-dimensional heterotic strings we have generically assumed the presence of just the dilaton tensor multiplet. This is in fact the correct description at a perturbative level. However, in general six-dimensional $N = 1$ theories more than one tensor multiplet may be present. Indeed, compactifications of M -theory on $K_3 \times S^1/Z_2$ leads to this possibility not seen at the perturbative level [20, 22, 25]. In fact, five-branes located at points (parametrized by five real coordinates) in this internal space will be generically present. A tensor and a hypermultiplet are associated to these branes. The five-branes are a source of torsion so that in a case with k_1 instantons in the first E_8 , k_2 in the second and $n_T - 1$ five-branes at points in $K_3 \times S^1/Z_2$, the condition $k_1 + k_2 = 24$ is replaced by

$$k_1 + k_2 + n_T - 1 = 24 \quad (2.16)$$

Here n_T is the number of tensor multiplets including that of the dilaton. Moreover, cancellation of gravitational anomalies leads to

$$n_H + 29n_T - n_V = 273 \quad (2.17)$$

This equation is for example satisfied by (2.4) since $n_T - 1 = 12 - n$.

Before getting into the specific discussion of 6d models either from the heterotic side or from an F-theory approach, let us recall that there are still heterotic/type II dual candidates that are not understandable from a six-dimensional point of view. The heterotic version of these models would require, in general, introduction of asymmetric orbifolds or enhancings involving the two-torus appearing when compactifying to $D = 4$ (or from $D = 10$ to $D = 8$, followed by a $K3$ compactification). This is the case for instance for the $(128, 2) \equiv \mathbb{P}_4^{(1,1,2,2,6)}$ [12] model discussed in [2]. Such models may still be organized into chains according to eq. (2.6), as was noticed in [7]. For example, $(76, 4) \rightarrow (99, 3) \rightarrow (128, 2)$. Also $(75, 9) \rightarrow (104, 8) \rightarrow (143, 7) \equiv \mathbb{P}_4^{(1,1,4,4,10)}$ [20] follows the same pattern. These cases do not correspond to elliptic fibrations. A possible scheme for obtaining the heterotic candidates some of these 4d chains is discussed at the end of next chapter.

3 Heterotic Strings on $K3$ and $K3 \times T^2$ Revisited

3.1 Non-semisimple Backgrounds

As we discussed in chapter 2, finding the heterotic duals of the chains of Calabi-Yau models of type B, C, ... seems to require new ingredients beyond the usual instanton embedding in $E_8 \times E_8$. In this chapter we will show how the desired new 6d heterotic models can be most readily obtained by considering generic $H \times U(1)^{8-d}$ backgrounds in each E_8 , with H some non-Abelian factor.

$U(1)$ backgrounds on $K3$ were first explored by Green, Schwarz and West [26]. The procedure can be applied to $SO(32)$ or $E_8 \times E_8$ heterotic strings. We will concentrate in the latter case and to begin we consider $U(1) \subset E_8$. The instanton number of the $U(1)$ configuration is defined to be

$$m_i = \frac{1}{16\pi^2} \int_{K3} \frac{1}{30} \text{Tr } F_{U(1)_i}^2 \quad ; \quad i = 1, 2 \quad (3.1)$$

Considering both E_8 's and imposing the requirement $\int_{K3} (\text{tr } R^2 - \frac{1}{30} \text{Tr } F^2) = 0$ gives the condition $m_1 + m_2 = 24$.

To determine the matter spectrum it is convenient to consider separately each E_8 broken to $E_7 \times U(1)$. The relevant adjoint decomposition is

$$\mathbf{248} = (\mathbf{133}, 0) + (\mathbf{56}, q) + (\mathbf{56}, -q) + (\mathbf{1}, 2q) + (\mathbf{1}, -2q) + (\mathbf{1}, 0) \quad (3.2)$$

Here $q = \frac{1}{2}$ so that the $U(1)$ generator Q is normalized as a generator of E_8 in the adjoint representation, *i.e.* $\text{Tr } Q^2 = 30$. According to the index theorem (2.1), the number of hypermultiplets of charge q is simply $N_q = mq^2 - 1$. Then,

$$\begin{aligned} N(\mathbf{56}, \frac{1}{2}) &= N(\mathbf{56}, -\frac{1}{2}) = \frac{m}{4} - 1 \\ N(\mathbf{1}, -1) &= N(\mathbf{1}, 1) = m - 1 \end{aligned} \quad (3.3)$$

Notice that to obtain positive multiplicities and half-integer number of $\mathbf{56}$'s, m must be an even integer with $m \geq 4$. Taking into account both E_8 's, the allowed values for (m_1, m_2) are $(24, 0)$, $(20, 4)$, $(18, 6)$, $(16, 8)$, $(14, 10)$ and $(12, 12)$. The hypermultiplet content of these $E_7 \times U(1) \times E_7 \times U(1)$ models is

$$\begin{aligned} &\left\{ \frac{1}{4}(m_1 - 4)(\mathbf{56}, \frac{1}{2}; \mathbf{1}, 0) + \frac{1}{4}(m_2 - 4)(\mathbf{1}, 0; \mathbf{56}, \frac{1}{2}) + \right. \\ &\left. (m_1 - 1)(\mathbf{1}, 1; \mathbf{1}, 0) + (m_2 - 1)(\mathbf{1}, 0; \mathbf{1}, 1) + \text{c.c.} \right\} + 20(\mathbf{1}, 0; \mathbf{1}, 0) \end{aligned} \quad (3.4)$$

where we have added the gravitational contribution.

It is easy to check that $U(1)$'s in this class of theories are in general anomalous. More precisely, one finds that the anomaly 8-form I_8 does not generically factorize into a product of two 4-forms so that the Green-Schwarz mechanism cannot cancel the residual anomaly. Instead one finds that the linear combination of $U(1)$ charges

$$Q_f = \cos \theta Q_1 + \sin \theta Q_2 \quad (3.5)$$

leads to a factorized I_8 as long as

$$\sin^2 \theta = \frac{m_2}{m_1 + m_2} \quad ; \quad \cos^2 \theta = \frac{m_1}{m_1 + m_2} \quad (3.6)$$

Thus, for given $m_{1,2}$, there is a linear combination of both $U(1)$'s which is anomaly-free but the orthogonal combination is not. Thus, somehow, the latter combination must be spontaneously broken. Indeed, a mechanism by which this can take place was suggested in refs. [27, 26] for analogous compactifications. The idea is that in $D = 10$ the kinetic term of the B_{MN} field contains a piece

$$H^2 \simeq (\partial_\mu B_{ij} + A_\mu^1 \langle F_{ij}^1 \rangle + A_\mu^2 \langle F_{ij}^2 \rangle)^2 \quad (3.7)$$

where the indices i, j live in the four compact dimensions. Notice that one linear combination of A_μ^1 and A_μ^2 will become massive by swallowing a B_{ij} zero mode. Specifically, since $\langle F_{ij}^a \rangle \simeq \sqrt{m_a}$ ($a = 1, 2$), precisely the orthogonal combination

to that in eq.(3.5) acquires a mass through this mechanism. Thus, the would-be anomalous $U(1)$ is in fact absent from the massless spectrum and the gauge group is actually $E_7 \times E_7 \times U(1)_f$. Notice that if we further break down $U(1)_f$ by giving vevs to some of the singlets in (3.4), the hypermultiplet content of this class of models with $U(1)$ instanton numbers (m_1, m_2) is analogous to that obtained with $SU(2)$ instantons (k_1, k_2) and k_i even. However, if the Higgs breaking proceeds through charged multiplets, at each stage of symmetry breaking there survives an unbroken $U(1)$ corresponding to a linear combination of Q_f and E_7 Cartan generators. We will see below that this residual $U(1)$ has an important role in understanding the extra families of models baptized B, C and D in the previous section.

Clearly, it is also possible to combine Abelian and non-Abelian backgrounds and have, for example, $H \times U(1)$ bundles with instanton numbers (k, m) inside each E_8 . The commutant is $G \times U(1)$ and the adjoint of E_8 decomposes as $\mathbf{248} = \sum_a (R_a, q_a, M_a)$ under $G \times U(1) \times H \subset E_8$. The number of hypermultiplets in the representation (R_a, q_a) of $G \times U(1)$ is again given by the index theorem

$$N(R_a, q) = kT(M_a) + mq^2 \dim M_a - \dim M_a \quad (3.8)$$

where we again normalize $\text{Tr } Q^2 = 30$. The generalization to $H \times U(1)^{8-d}$, $d \leq 6$, is straightforward. In the following we will consider various choices leading to the type B, C and D heterotic duals.

3.1.1 Type B Models

Choosing $SU(2) \times U(1)$ as background gives an unbroken subgroup $E_6 \times U(1)$ arising from each E_8 . The relevant adjoint decomposition is

$$\begin{aligned} \mathbf{248} = & (\mathbf{78}, 0, \mathbf{1}) + (\mathbf{1}, 0, \mathbf{1}) + (\mathbf{1}, 0, \mathbf{3}) + (\mathbf{27}, q, \mathbf{2}) + (\overline{\mathbf{27}}, -q, \mathbf{2}) + \\ & (\overline{\mathbf{27}}, 2q, \mathbf{1}) + (\mathbf{27}, -2q, \mathbf{1}) + (\mathbf{1}, 3q, \mathbf{2}) + (\mathbf{1}, -3q, \mathbf{2}) \end{aligned} \quad (3.9)$$

where now $q = \frac{1}{2\sqrt{3}}$. Embedding $SU(2) \times U(1)$ backgrounds with instanton numbers $(k_1, m_1; k_2, m_2)$ in both E_8 's gives the following $E_6 \times U(1) \times E_6 \times U(1)$ spectrum

$$\begin{aligned} & \left\{ \frac{1}{6}(3k_1 + m_1 - 12)(\mathbf{27}, \frac{1}{2\sqrt{3}}; \mathbf{1}, 0) + \frac{1}{6}(3k_2 + m_2 - 12)(\mathbf{1}, 0; \mathbf{27}, \frac{1}{2\sqrt{3}}) + \right. \\ & \left. \frac{1}{3}(m_1 - 3)(\mathbf{27}, -\frac{1}{\sqrt{3}}; \mathbf{1}, 0) + \frac{1}{3}(m_2 - 3)(\mathbf{1}, 0; \mathbf{27}, -\frac{1}{\sqrt{3}}) + \right. \end{aligned}$$

$$\frac{1}{2}(k_1 + 3m_1 - 4)(\mathbf{1}, \frac{\sqrt{3}}{2}; \mathbf{1}, 0) + \frac{1}{2}(k_2 + 3m_2 - 4)(\mathbf{1}, 0; \mathbf{1}, \frac{\sqrt{3}}{2}) + \text{c.c.} \} + ((2k_1 - 3) + (2k_2 - 3) + 20)(\mathbf{1}, 0; \mathbf{1}, 0) \quad (3.10)$$

In this case gravitational anomalies cancel as long as $k_1 + m_1 + k_2 + m_2 = 24$. Again, we find that, independently of the values of $k_{1,2}$, the linear combination in eq.(3.5) leads to a factorized $U(1)$ anomaly that can be cancelled by the GS mechanism. The orthogonal linear combination is expected to be Higgsed away as in the previous case.

Notice that in the presence of the $SU(2)$ bundles the values of $m_{1,2}$ are forced to be multiples of 3 in order to have half-integer numbers of $(\mathbf{27} + \overline{\mathbf{27}})$ and also $m_{1,2} \geq 3$. Thus the simplest class of models of this type will have instanton numbers $(k_1, 3; k_2, 3)$ and the unbroken $U(1)_f$ is in this case the diagonal combination $U(1)_D$. The $E_6 \times E_6 \times U(1)_D$ spectrum is then given by

$$\left\{ \frac{1}{2}(k_1 - 3)(\mathbf{27}, \mathbf{1}, \frac{1}{2\sqrt{6}}) + \frac{1}{2}(k_2 - 3)(\mathbf{1}, \mathbf{27}, \frac{1}{2\sqrt{6}}) + \frac{1}{2}(k_1 + k_2 + 10)(\mathbf{1}, \mathbf{1}, \frac{\sqrt{3}}{2\sqrt{2}}) + \text{c.c.} \right\} + (2k_1 + 2k_2 + 13)(\mathbf{1}, \mathbf{1}, 0) \quad (3.11)$$

This matter content has anomaly polynomial

$$I_8 = (\text{tr } R^2 - \frac{1}{3}\text{tr } F_1^2 - \frac{1}{3}\text{tr } F_2^2 - f^2)(\text{tr } R^2 - \frac{k_1 - 9}{6}\text{tr } F_1^2 - \frac{k_2 - 9}{6}\text{tr } F_2^2 - 3f^2) \quad (3.12)$$

where F_i is the field strength of the i -th E_6 and f is that of $U(1)_D$. Notice that the mixed $\text{tr } F_1^2 \text{tr } F_2^2$ term vanishes by virtue of the constraint $k_1 + k_2 = 18$.

The fact that $k_1 + k_2 = 18$, instead of $k_1 + k_2 = 24$ in the case without $U(1)$ backgrounds, hints at the required heterotic duals of models of type B. Indeed, in these models, the range for the values of n is smaller ($n \leq 8$) and this is probably the case here since the range for $k_{1,2}$ is also smaller. Moreover, models B have a number of vector multiplets one unit higher compared to the corresponding chain A elements. This is precisely the case here, due to the presence of the extra $U(1)_D$. These arguments are compelling enough to consider this sort of heterotic constructions in more detail. We will see that upon sequential Higgsing of the non-Abelian symmetries the spectrum in (3.11) does in fact reproduce chains of type B.

In analogy with the usual situation, we will label the models in terms of the integer

$$n = k_1 + m_1 - 12 \quad (3.13)$$

where we assume without loss of generality that $k_1 + m_1 \geq 12$. We choose $m_1 = m_2 = 3$ as before so that $k_1 + k_2 = 18$ (we will show that for $n = 7$ the k_2 instantons become small). We now set up the derivation of the spectrum implied by (3.11) upon maximal Higgsing of non-Abelian symmetries. The results of course depend on n or equivalently on the pair (k_1, k_2) . The strategy is to first implement breaking of the second E_6 together with $U(1)_D$ to $G_0 \times U(1)_X$, where $U(1)_X$ is the appropriate ‘skew’ combination of $U(1)_D$ and an E_6 Cartan generator. Since $k_1 \geq 9$, the first E_6 together with $U(1)_X$ can then be broken to another ‘skew’ $U(1)_Y$. The terminal gauge group is therefore $G_0 \times U(1)_Y$ which by construction has a factorized anomaly polynomial. Except for $n = 5$, the terminal matter consists of G_0 singlets charged under $U(1)_Y$ plus a number of completely neutral hypermultiplets. The final step is to perform a toroidal compactification on T^2 followed by transition to the Coulomb phase. This allows us to compare the resulting spectrum of vector and hypermultiplets with the Hodge numbers of candidate dual type II compactifications. It is also interesting to consider un-Higgsings in the first E_6 along different branches. We will study in particular un-Higgsing of $SU(r)$ factors in order to identify the type B chains more precisely. We now sketch the outcome for the different allowed values of n .

$n = 0, 1, 2$

In these cases $(k_1, k_2) = (9, 9), (10, 8)$ and $(11, 7)$ respectively and the terminal group is just $U(1)_Y$. The terminal hypermultiplet content is

$$\{48(\frac{1}{2\sqrt{2}}) + \text{c.c.}\} + 149(0) \quad (3.14)$$

in all three cases. In the 4d Coulomb phase there are then 4 vector multiplets and 149 massless hypermultiplets. This implies Hodge numbers $(b_{21}, b_{11}) = (148, 4)$ in agreement with the values for B models given in Table 2. From the spectrum (3.14) it is obvious that if we further break $U(1)_Y$ we end up with 244 hypermultiplets and no 6d vector multiplets, corresponding to the final elements of the $n = 0, 1, 2$ A chains. Although these three cases yield similar spectra after full Higgsing of $E_6 \times E_6$, if we un-Higgs in steps it is easy to see that they behave differently. For example, un-Higgsing an $SU(2)$ factor in the first E_6 gives $SU(2) \times U(1)_Y$ spectrum

$$\{(3n + 8)(\mathbf{2}, \frac{1}{2\sqrt{7}}) + (41 - 3n)(\mathbf{1}, \frac{1}{\sqrt{7}}) + \text{c.c.}\} + (134 - 6n)(\mathbf{1}, 0) \quad (3.15)$$

The number of vector and hypermultiplets in the 4d Coulomb phase clearly matches the Hodge numbers given in (2.11).

$n = 3$

Here $(k_1, k_2) = (12, 6)$ and the terminal group is $SU(3) \times U(1)_Y$ with hypermultiplets transforming as

$$\{50(\mathbf{1}, \frac{1}{2}\sqrt{\frac{3}{5}}) + \text{c.c.}\} + 153(\mathbf{1}, 0) \quad (3.16)$$

Upon toroidal compactification and transition to the 4d Coulomb phase the spectrum matches the Hodge numbers $(b_{21}, b_{11}) = (152, 6)$ characteristic of the space $\mathbb{P}_4^{(1,1,3,5,10)}$ [20]. Notice again that Higgsing $U(1)_Y$ leads to the final element of the $n = 3$ type A chain with $(b_{21}, b_{11}) = (251, 5)$.

$n = 4$

In this case $(k_1, k_2) = (13, 5)$ and the terminal group is $SO(8) \times U(1)_Y$ with hypermultiplet content

$$\{54(\mathbf{1}, \frac{1}{\sqrt{6}}) + \text{c.c.}\} + 165(\mathbf{1}, 0) \quad (3.17)$$

In the 4d Coulomb phase we obtain a spectrum with type IIA dual characterized by $(b_{21}, b_{11}) = (164, 8)$. These are the Hodge numbers of $\mathbb{P}_4^{(1,1,4,6,12)}$ [24]. Observe the presence of an enhanced $SO(8)$ group just as it happens in the $n = 4$ type A terminal model.

$n = 5$

In this case we have $(k_1, k_2) = (14, 4)$ and upon Higgsing we arrive at a gauge group $E_6 \times U(1)_Y$ with hypermultiplets transforming as

$$\{\frac{1}{2}(\mathbf{27}, \frac{1}{4\sqrt{3}}) + \frac{117}{2}(\mathbf{1}, \frac{\sqrt{3}}{4}) + \text{c.c.}\} + 179(\mathbf{1}, 0) \quad (3.18)$$

Due to the $U(1)$ charge, the gauge symmetry cannot be further broken (as long as the $U(1)$ remains unbroken). Going to the 4d Coulomb phase we arrive at a model corresponding to Hodge numbers $(b_{21}, b_{11}) = (178, 10)$.

$n = 6$

Here $(k_1, k_2) = (15, 3)$ so that the second E_6 has no charged matter and cannot be broken. The terminal group is then $E_6 \times U(1)_Y$ with hypermultiplet content

$$\{64(\mathbf{1}, \frac{\sqrt{3}}{4}) + \text{c.c.}\} + 195(\mathbf{1}, 0) \quad (3.19)$$

Recall that the $n = 6$ type A chain has a matter-free E_6 as terminal group. In the case at hand, going to the 4d Coulomb phase implies a dual with $(b_{21}, b_{11}) = (194, 10)$ in agreement with the Hodge numbers of $\mathbb{P}_4^{(1,1,6,8,16)}$ [32].

$n = 7$

Naively we would set $k_2 = 2$ but this does not lead to sensible results as it is obvious from eq.(3.11). We will then remove the k_2 $SU(2)$ instantons so that we are left with just an $U(1)$ background in the second E_8 . From eq. (3.3) we see that we need to have $m_2 \geq 4$, m_2 even. Hence, one of the two instantons in k_2 is employed in raising m_2 from 3 to 4 and the other becomes point-like. We then have a distribution $(k_1, m_1; k_2, m_2) = (16, 3; 0, 4)$, plus a point-like instanton giving rise to a tensor multiplet. Cancellation of gravitational anomalies is guaranteed since $k_1 + m_1 + m_2 + 1 = 24$, where the extra 1 is due to the presence of an M-theory 5-brane (which in term gives rise to a 6d tensor multiplet). The E_7 arising from the second E_8 has no charged matter as implied by (3.3). Then, the terminal group is $E_7 \times U(1)_Y$ with hypermultiplets transforming as

$$\{69(\mathbf{1}, \frac{1}{\sqrt{5}}) + \text{c.c.}\} + 210(\mathbf{1}, 0) \quad (3.20)$$

To these we must add one neutral hypermultiplet and one tensor multiplet whose 5 scalar components parametrize the position of the 5-brane in $K3 \times S^1/Z_2$. Upon further toroidal compactification the tensor multiplet gives rise to a 4d vector multiplet. Moving to the 4d Coulomb branch we land on a model that matches $(b_{21}, b_{11}) = (210, 12)$. These are the Hodge numbers of $\mathbb{P}_4^{(1,1,7,9,18)}$ [36].

$n = 8$

We now have $(k_1, m_1; k_2, m_2) = (17, 3; 0, 4)$ and, unlike the previous situation, there is no small instanton. The gauge group upon Higgsing is $E_7 \times U(1)_Y$ with hypermultiplets transforming as

$$\{75(\mathbf{1}, \frac{1}{\sqrt{5}}) + \text{c.c.}\} + 228(\mathbf{1}, 0) \quad (3.21)$$

In the 4d Coulomb branch we find a model dual to a type IIA compactification on a CY with $(b_{21}, b_{11}) = (227, 11)$. These are precisely the Hodge numbers of $\mathbb{P}_4^{(1,1,8,10,20)}$ [40], the last element of the $n = 8$ type B chain.

It is now easy to understand from the heterotic side why $n \leq 8$. The next element in the series would have $k_1 = 18$ and $m_1 = 3$. For the second E_8 the only sensible

alternative consistent with anomaly cancellation is to have 3 point-like instantons. The terminal gauge group is then $E_8 \times U(1)_Y$ with hypermultiplets transforming as

$$\{78(\mathbf{1}, \frac{1}{2}) + \text{c.c.}\} + 246(\mathbf{1}, 0) \quad (3.22)$$

There are in addition 3 tensor multiplets and 3 hypermultiplets whose scalar components parametrize the positions of the three 5-branes. In this situation the $U(1)$ is actually anomalous and it is thus Higgsed away. Therefore, we are left altogether with 405 hypermultiplets. In four dimensions the enhanced gauge symmetry is $E_8 \times U(1)_{\text{tensor}}^3 \times U(1)^4$, implying a Coulomb branch with 14 vector multiplets and 405 hypermultiplets. The dual then corresponds to a type IIA compactification on a CY with Hodge numbers (404,14). This is nothing but the final element of the $n = 9$ type A chain. Hence, type B chains stop at $n = 8$ because for $n \geq 9$ they fall back into type A chains.

The foregoing heterotic construction not only matches the Hodge numbers of the last elements of chains of type B but also reproduces the preceding elements in each of the chains. Indeed, considering the symmetry breaking sequence $SU(3) \times U(1) \rightarrow SU(2) \times U(1) \rightarrow U(1)$, and going to the 4d Coulomb phase, we find results in complete agreement with (2.11) for all n . We can also consider other un-Higgsing patterns. For example, for $SU(4) \times U(1)$ the corresponding Hodge numbers are given by

$$b_{12}^4 = b_{12}^1 - (12n + 33) \quad ; \quad b_{11}^4 = b_{11}^1 + 3 \quad (3.23)$$

Notice also that chains of type A and B are connected in the heterotic side by Higgsing of the $U(1)$ present in the latter. Thus, at each step of chain B there is a Higgs branch connecting it to the corresponding step in the type A chain with same n .

3.1.2 Type C and D Models

We now consider $SU(2) \times U(1)^2$ backgrounds in each E_8 . The $U(1)$'s are embedded according to the branchings $E_8 \supset SO(10) \times SU(4)$ and $SU(4) \supset SU(2) \times SU(2)_A \times U(1)_B \supset SU(2) \times U(1)_A \times U(1)_B$. The distribution of instanton numbers is chosen to be $(k_1, m_{1A}, m_{1B}; k_2, m_{2A}, m_{2B}) = (k_1, 3, 2; k_2, 3, 2)$, which can be shown to guarantee a consistent spectrum. Notice that anomaly cancellation requires $k_1 + k_2 = 14$ (in the absence of extra tensor multiplets from small instantons). The unbroken gauge group at the starting level is $SO(10) \times U(1)^2 \times SO(10) \times U(1)^2$. In this case the diagonal

combinations $Q_{AD} = \frac{1}{\sqrt{2}}(Q_{1A} + Q_{2A})$ and $Q_{BD} = \frac{1}{\sqrt{2}}(Q_{1B} + Q_{2B})$ are anomaly-free whereas their orthogonal combinations are anomalous and are expected to be Higgsed away by a mechanism analogous to that explained before. The $SO(10) \times SO(10) \times U(1)_{AD} \times U(1)_{BD}$ hypermultiplets in the massless spectrum are

$$\begin{aligned} & \left\{ \frac{1}{2}(k_1 - 3)(\mathbf{16}, \mathbf{1}, 0, -\frac{1}{4}) + \frac{1}{2}(k_2 - 3)(\mathbf{1}, \mathbf{16}, 0, -\frac{1}{4}) + \frac{1}{2}(k_1 - 1)(\mathbf{10}, \mathbf{1}, \frac{1}{2\sqrt{2}}, 0) + \right. \\ & \frac{1}{2}(k_2 - 1)(\mathbf{1}, \mathbf{10}, \frac{1}{2\sqrt{2}}, 0) + \frac{1}{2}(k_1 + 3)[(\mathbf{1}, \mathbf{1}, \frac{1}{2\sqrt{2}}, -\frac{1}{2}) + (\mathbf{1}, \mathbf{1}, \frac{1}{2\sqrt{2}}, \frac{1}{2})] + \\ & 4(\mathbf{1}, \mathbf{1}, \frac{1}{\sqrt{2}}, 0) + \frac{1}{2}(k_2 + 3)[(\mathbf{1}, \mathbf{1}, \frac{1}{2\sqrt{2}}, -\frac{1}{2}) + (\mathbf{1}, \mathbf{1}, \frac{1}{2\sqrt{2}}, \frac{1}{2})] + \text{c.c.} \} + \\ & (2k_1 + 2k_2 + 12)(\mathbf{1}, \mathbf{1}, 0, 0) \end{aligned} \quad (3.24)$$

where we have included the gravitational contribution.

Since we are setting $m_{1A} = m_{2A} = 3$ and $m_{1B} = m_{2B} = 2$, the possible choices for the $SU(2)$ instanton numbers are $(k_1, k_2) = (7, 7), (8, 6), (9, 5), (10, 4)$, and $(11, 3)$. It is again convenient to label the models in terms of the integer

$$n = k_1 + m_{1A} + m_{1B} - 12 \quad (3.25)$$

To identify the terminal elements for each n we implement breaking to $G_0 \times U(1)_Y \times U(1)_Z$, where G_0 arises from the second $SO(10)$ and the surviving $U(1)$'s are the appropriate oblique combinations of $U(1)_{AD}$, $U(1)_{BD}$ and $SO(10) \times SO(10)$ Cartan generators. Let us briefly discuss the main features of the models for the different values of n .

$n = 0, 1, 2$

In these cases $(k_1, k_2) = (7, 7), (8, 6), (9, 5)$ respectively. The terminal group is $U(1)_Y \times U(1)_Z$ with spectrum consisting of 144 charged and 102 neutral hypermultiplets in all three cases. In the 4d Coulomb phase we find a model that matches $(b_{21}, b_{11}) = (101, 5)$, in agreement with the Hodge numbers given in Table 2. Un-Higgsing of an $SU(2)$ factor shows different spectra for different n and gives corresponding Hodge numbers in accord with (2.11). By Higgsing sequentially $U(1)_Y \times U(1)_Z \rightarrow U(1)_Y \rightarrow \emptyset$, we obtain the corresponding last elements of the $n = 0, 1, 2$ chains of type B and A.

$n = 3$

Here $(k_1, k_2) = (10, 4)$. The terminal group is $SU(3) \times U(1)_Y \times U(1)_Z$ with 104 charged and 150 neutral hypermultiplets. In the 4d Coulomb phase we obtain a

model with corresponding $(b_{21}, b_{11}) = (103, 7)$. These are the Hodge numbers of $\mathbb{P}_4^{(1,1,3,5,5)}$ [15].

$n = 4$

In this case $(k_1, k_2) = (11, 3)$ and the second $SO(10)$ can be broken to a matter-free $SO(8)$. Maintaining $U(1)_Y \times U(1)_Z$ also unbroken, we arrive at a model with 112 singlet hypermultiplets plus extra charged matter. In the 4d Coulomb branch we find $(b_{21}, b_{11}) = (111, 9)$. This matches the Hodge numbers of $\mathbb{P}_4^{(1,1,4,6,6)}$ [18], which is the last element of the $n = 4$ type C chain. Notice that we again have an unbroken $SO(8)$ symmetry as it happens in chains of type A and B.

$n = 5$

In this case we would naively set $k_1 = 12$ and $k_2 = 2$, but the latter is not possible since $k_2 = 2$ could not support an $SU(2)$ bundle. The appropriate distribution of instantons turns out to be $(k_1, m_{1A}, m_{1B}; k_2, m_{2A}, m_{2B}) = (12, 3, 2; 0, 3, 3)$. Thus, one of the two instantons from the removed $SU(2)$ is used to increase m_{2B} from 2 to 3 (this is required to avoid inconsistencies in the spectrum) whereas the other instanton becomes small giving rise to an extra 6d tensor multiplet. This is consistent with cancellation of anomalies, *i.e.* $k_1 + m_{1A} + m_{1B} + m_{2A} + m_{2B} + 1 = 24$, where the extra unit comes from the contribution of the small instanton. The resulting model has terminal group $E_6 \times U(1)_Y \times U(1)_Z$ plus one tensor multiplet and a neutral hypermultiplet. In the 4d Coulomb phase we find 12 vector multiplets (one coming from the 6d tensor multiplet) and 121 hypermultiplets. This would correspond to a type IIA compactification on a CY with $(b_{21}, b_{11}) = (120, 12)$. These are the Hodge numbers of $\mathbb{P}_4^{(1,1,5,7,7)}$ [21].

$n = 6$

The instanton assignments are $(k_1, m_{1A}, m_{1B}; k_2, m_{2A}, m_{2B}) = (13, 3, 2; 0, 3, 3)$. The small instanton in the prior situation has travelled to the first E_8 and acquired a finite size so there are no extra tensor multiplets. The terminal group is again $E_6 \times U(1)_Y \times U(1)_Z$, with 132 singlet hypermultiplets plus extra E_6 singlets charged under the $U(1)$'s. In the 4d Coulomb branch we encounter a model with 11 vector multiplets and 132 hypermultiplets. This corresponds to the Hodge numbers of $\mathbb{P}_4^{(1,1,6,8,8)}$ [24], the last element of the $n = 6$ C chain, as expected.

Again, the preceding heterotic construction also predicts the associated Hodge numbers for many un-Higgsing patterns. For $SU(2) \times U(1)^2$ we find results in accord with (2.11) for all n . For $SU(3) \times U(1)^2$ and $SU(4) \times U(1)^2$ we obtain

$$\begin{aligned} b_{12}^4 &= b_{12}^1 - (8n + 25) & ; & & b_{11}^4 &= b_{11}^1 + 3 \\ b_{12}^3 &= b_{12}^1 - (7n + 20) & ; & & b_{11}^3 &= b_{11}^1 + 2 \end{aligned} \quad (3.26)$$

Other branches emanating from or leading to $SO(10) \times U(1)^2$ can be followed as well.

Notice that, in principle, there is no obstruction to the inclusion of more Abelian background factors, and therefore to the existence of new types of models. We only expect to have a shorter range for n since the extra factors soak up more instanton numbers and, as observed before, higher values of n will fall into chains already present. For instance, D models can be obtained by including $SU(2) \times U(1)^3$ backgrounds, with $n = 0, 1, 2, 3, 4$ given by the straightforward generalization of (3.25). As an illustration, notice that in the $n = 0$ case, with equal instanton numbers in each $SU(2)$ factor, the initial $SU(5) \times SU(5)$ non-Abelian gauge group can be completely Higgsed away. This corresponds to a deformation of the initial background to $SO(10) \times U(1)^3$ with nine instantons in each $SO(10)$. The model obtained has Hodge numbers $(70, 6)$.

3.2 Semisimple Backgrounds

Interesting possibilities open when semisimple non-Abelian backgrounds, instead of the simple H factors included so far, are allowed. In particular, (R, \bar{R}) representations, leading to higher Kac-Moody level groups with adjoint matter, can naturally appear. We now want to show how an alternative construction for some chains can be achieved in this manner. We use a notation in which subscripts between parentheses denote instanton numbers whereas plain subscripts indicate the Kac-Moody level.

As an example, consider an $SU(2)_{(8)} \times SU(2)_{(6)}$ semisimple bundle with instanton numbers $(8, 6)$ in the first E_8 and an $SU(2)_{(10)}$ bundle with ten instantons in the second. The observable group is $SO(12) \times E_7$. Now we Higgs E_7 away and break down $SO(12)$ to $SU(6)$ (which could also be obtained by embedding an $SU(2)_{(8)} \times SU(3)_{(6)}$ bundle). Breaking $SU(6)$ to $SU(5)$ and then continuing along $SU(4) \rightarrow$

$SU(3) \rightarrow SU(2) \rightarrow \emptyset$, we recover the $n = 2$ type A chain. This process can be seen as a deformation of the original $SU(2) \times SU(2)$ through simple group bundles. Alternatively, one can proceed by maintaining the semisimple structure, *e.g.* by breaking $SU(6)$ to $SU(3) \times SU(3)$ from the Higgsing point of view. In this way we arrive at a spectrum

$$12[(\mathbf{3}, \mathbf{1}) + (\bar{\mathbf{3}}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{1}, \bar{\mathbf{3}})] + (\mathbf{3}, \bar{\mathbf{3}}) + (\bar{\mathbf{3}}, \mathbf{3}) + \mathbf{98}(\mathbf{1}, \mathbf{1}) \quad (3.27)$$

This $SU(3) \times SU(3)$ can also be derived using a $SU(3)_{(8)} \times SU(3)_{(6)}$ background. Notice the presence of $(\mathbf{3}, \bar{\mathbf{3}}) + (\bar{\mathbf{3}}, \mathbf{3})$ representations that can effect the breaking to the diagonal $SU(3)_2$ at level two. Along the direction $SU(3)_2 \rightarrow SU(2)_2 \rightarrow \emptyset$ we now encounter matching Hodge numbers $(101, 5) \rightarrow (148, 4) \rightarrow (243, 3)$, corresponding to $C \rightarrow B \rightarrow A$ for $n = 2$. Yet another alternative is to take a different route breaking $U(1)$ subgroups in each $SU(3)$ to the diagonal combination to arrive at $SU(2) \times SU(2) \times U(1)$. Diagonal Higgsing then leads to $SU(2) \times U(1)$ and finally $U(1)$. In this way we reproduce the $n = 2$ type B chain $(98, 6) \rightarrow (121, 5) \rightarrow (148, 4)$. This whole sequence corresponds to a deformation of the starting $SU(2)_{(8)} \times SU(2)_{(6)}$ bundle through $SU(4)_{(8)} \times SU(3)_{(6)} \rightarrow SU(7)_{(14)} \rightarrow SO(14)_{(14)}$. Notice again the existence of an $U(1)$ Abelian factor generated here through a seemingly different procedure. Finally, let us stress that, in spite of this alternative construction for $n = 2$, we have only been able to obtain a unified picture for all chains by considering the $U(1)$ backgrounds studied in the previous subsections.

The inclusion of semisimple non-Abelian bundles also furnishes a possible explanation of the 4d chains mentioned at the end of last chapter. Let us examine for instance the chain ending at $(143, 7)$. As discussed, these models are expected to be originated in a compactification involving enhancing of toroidal $U(1)$'s. We then perform a toroidal compactification down to eight dimensions adjusting moduli parameters in order to obtain an $SO(20) \times E_8$ gauge group as in example 10 of ref. [2]. The next step is a further compactification on $K3$ down to four dimensions, with an $SU(2)_{(6)} \times SU(2)_{(10)}$ semisimple bundle with $(k_1, k_2) = (6, 10)$ instantons embedded in $SO(20)$ and an $SU(2)_{(8)}$ bundle with $k_3 = 8$ in E_8 . The starting gauge group is therefore $SO(12) \times SU(2) \times SU(2) \times E_7$ (not including dilaton and graviphoton). E_7 can be Higgsed down to a terminal $SO(8)$ without charged matter. This breaking corresponds to a deformation of the original $SU(2)_{(8)}$ to an $SO(8)_{(8)}$ bundle. Higgsing the other factors is more subtle and we proceed as before by systematically

breaking to the diagonal $U(1)$ built up from Abelian subgroups contained in different factors. The last steps correspond to continuous deformations of the original bundle according to

$$SU(3)_{(6)} \times SU(5)_{(10)} \rightarrow SU(8)_{(16)} \rightarrow SO(16)_{(16)} \quad (3.28)$$

Let us check that these bundles do indeed produce the Hodge numbers that we are looking for. To this purpose we need to compute the dimensions $\dim \mathcal{M}_k(H)$ of the moduli space associated to each bundle H with instanton number k . Equivalently, we need to determine the number of neutral singlets arising from each H . From the index theorem (2.1) we easily obtain

$$\dim \mathcal{M}_k(H) = kc_H - \dim H \quad (3.29)$$

where $c_H = T(\text{adj } H)$ is the Coxeter number of H . Hence, $\dim \mathcal{M}_8(SO(8)) = 20$, $\dim \mathcal{M}_{(6,10)}(SU(3) \times SU(5)) = 36$, etc. Adding the 20 gravitational moduli we thereby obtain the 4d chain of Hodge numbers $(75, 9) \rightarrow (104, 8) \rightarrow (143, 7)$.

4 Type II Compactifications

4.1 F-theory Duals of the New Heterotic Models

Recently a new insight into several string dualities has been provided by F-theory [4], a construction that can be understood as a type IIB compactification on a variety B in the presence of Dirichlet 7-branes. The complex ‘coupling constant’ $\tau = a + ie^{-\varphi/2}$, where a is the RR scalar field and φ is the dilaton field, depends on space-time and is furthermore allowed to undergo $SL(2, Z)$ monodromies around the 7-branes. This τ can be thought to describe the complex structure parameter of a torus (of frozen Kähler class, since type IIB theory has no fields to account for it) varying over the compactifying space B , and degenerating at the 8d submanifolds defined by the world-volumes of the 7-branes. The constraint of having vanishing total first Chern class (the contribution of the 7-branes cancelling that of the manifold B) forces the T^2 over B fibration thus constructed to be an elliptic CY manifold X . Thus, F-theory can be understood as a 12d construction which has consistent compactifications on elliptically fibered manifolds. It has been conjectured [4] that F-theory compactified on the product of such an elliptically fibered manifold X and

a circle S^1 , lies on the same moduli space as M-theory compactified on X . This idea has proved fruitful in encoding string dualities in lower dimensions, and, especially, in clarifying several phenomena in heterotic string compactifications.

After compactification on an elliptic $K3$, F-theory gives an 8d theory conjectured to be dual to the heterotic string compactified on T^2 [4, 6]. When the elliptic fiber is chosen to be $\mathbb{P}_2^{(1,2,3)}$ [6], the (1,1) $K3$ moduli and the heterotic moduli are related as follows. The size of the base \mathbb{P}_1 is related to the heterotic dilaton whereas the 18 polynomial deformation complex parameters of the fibration match the heterotic toroidal Kähler and complex structure moduli together with Wilson line backgrounds. The Kähler class of the A_1 singularity of this particular $K3$ is associated to the size of the fiber, it has no physical meaning in F-theory, and thus, no heterotic counterpart.

Fibered this model over another \mathbb{P}_1 gives a family of F-theory compactifications on CY 3-folds which are $K3$ fibrations with the $K3$ fibers admitting an elliptic fibration structure. The resulting base spaces are the Hirzebruch surfaces \mathbb{F}_n , which are fibrations of \mathbb{P}_1 over \mathbb{P}_1 , characterized by a non-negative integer n . These models are naturally conjectured to be dual to heterotic string compactifications on $K3$ (T^2 fibered over \mathbb{P}_1) with gauge bundles embedded on $E_8 \times E_8$ (for some values of n , it can also be related to $SO(32)$ heterotic string compactifications [28, 29]). Upon toroidal compactification to $D = 4$, $N = 2$ heterotic/type II duality is recovered so that an $N = 1$, $D = 6$ version of this duality is actually introduced. This has several advantages, as dynamics in six dimensions are quite constrained, and have been under detailed study in recent works. Concerning this issue, let us note that the base space of these compactifications, \mathbb{F}_n , has two Kähler forms, and thus the massless spectrum contains only one tensor multiplet (associated to the heterotic dilaton [5]). Consequently, except when the singularities in the variety require a blow-up of the base for their resolution, we will have $n_T = 1$.

Our purpose in this section is to find the F-theory duals of the previously discussed heterotic models. We will use the very detailed work of ref. [5] as a guide. We briefly review some of the main results in order to fix the notation and stress the analogies among the duals of the different types of heterotic chains. The choice of specific elliptic fiber $\mathbb{P}_2^{(1,2,3)}$ [6] implies CY spaces that can be described as follows. Introducing variables z_1, w_1 and z_2, w_2 for the two \mathbb{P}_1 's, x, y for the torus, and two

\mathbb{C}^* quotients to projectivize the affine spaces, gives the structure

$$\begin{array}{ccccccc}
& z_1 & w_1 & z_2 & w_2 & x & y \\
\lambda & 1 & 1 & 0 & 0 & 4 & 6 \\
\mu & n & 0 & 1 & 1 & 2n+4 & 3n+6
\end{array} \tag{4.1}$$

The hypersurface in this space is given by the fibration equation

$$y^2 = x^3 + f(z_1, w_1; z_2, w_2)x + g(z_1, w_1; z_2, w_2) \tag{4.2}$$

It can be shown that for $n > 12$ the variety described by (4.1) does not fulfill the CY condition (in particular, the associated Newton polyhedron ceases to be reflexive), so that there are 13 possible spaces.

For $n \neq 0, 1$, we can dehomogenize with respect to w_1 using one of the \mathbb{C}^* quotients, and the variety can be represented by the hypersurface $\mathbb{P}_4^{(1,1,n,2n+4,3n+6)}[6n+12]$. These coincide with the last elements of the chains of A models. Furthermore, for all values of n , the Hodge numbers of the fibration do match with the matter spectrum of heterotic models on $K3 \times T^2$ with $SU(2)$ bundles of $(12+n, 12-n)$ instanton number embedded in $E_8 \times E_8$, upon maximal Higgsing and moving to the Coulomb phase [7, 10]. Thus, one identifies type IIA compactifications on these spaces with duals of the heterotic constructions in $D = 4$, or equivalently the F-theory compactifications with the heterotic models in $D = 6$ (decompactifying the T^2).

The $E_8 \times E_8$ structure in the F-theory compactification can be deduced by analyzing the defining equation (4.2) near the regions $z_1 = 0$ and $z_1 = \infty$ [5]. We illustrate this strategy since it will be of constant reference along this section. To this end we expand the polynomials f, g in powers of z_1, w_1

$$\begin{aligned}
f(z_1, w_1; z_2, w_2) &= \sum_{k=-4}^4 z_1^{4+k} w_1^{4-k} f_{8-nk}(z_2, w_2) \\
g(z_1, w_1; z_2, w_2) &= \sum_{l=-6}^6 z_1^{6+l} w_1^{6-l} g_{12-nl}(z_2, w_2)
\end{aligned} \tag{4.3}$$

where subscripts denote the degree of the polynomial in z_2 (only non-negative degrees are admitted).

A first check [5] of duality consists of the identification of the generic type of singular fiber along the curves $z_1 = 0$ and $z_1 = \infty$ in the base. The singularity type is associated to the terminal gauge group after maximal Higgsing in the heterotic

side. Also, some information about the hypermultiplet content of the theory can be obtained [5, 12, 30]. The existence of hypermultiplets charged under the gauge group is detected by the worsening of the singularities as z_2 varies. It is possible to calculate the numbers of singlets coming from each heterotic E_8 bundle by counting the number of monomial deformations (modulo redundancies) near $z_1 = 0$ (polynomials of type ‘ f ’ in (4.3) with $k < 0$, and of type ‘ g ’ with $l < 0$), and near $z_1 = \infty$ (polynomials with $k, l > 0$). It is interesting to note that the outcome of this calculation gives a contribution of $[30(12+n)-248]$ and $[30(12-n)-248]$ singlets for the case $n \leq 2$ when complete Higgsing of the gauge group is possible. This is a signal of the already mentioned heterotic $E_8 \times E_8$ bundle structure with $(12+n, 12-n)$ instantons. The remaining polynomials ($k, l = 0$) give the 20 hypermultiplets associated to the gravitational contribution of the heterotic $K3$ compactification. A third check of the equivalence of the heterotic and F-theory description of the models comes from the construction of CY spaces dual to heterotic models with perturbatively enhanced gauge groups, and with non-perturbative effects. In this way, one can reproduce a web of F-theory models matching that obtained in the heterotic approach [10, 12].

The interplay of F-theory and M-theory in the study of 6d heterotic dynamics has been remarkably useful in checking different ideas about phase transitions, and new Higgs branches. This motivates the search for the F-theory duals of the heterotic constructions presented in the preceding section. Our analysis for models B, C and D is not as detailed as that already performed in the literature for models A, but it provides enough evidence to strongly suggest the nature of the F-theory duals of the different models and their connections.

The basic idea is to repeat the previous arguments using as elliptic fibers other torus realizations such as $\mathbb{P}_2^{(1,1,2)}[4]$, $\mathbb{P}_2^{(1,1,1)}[3]$, $\mathbb{P}_3^{(1,1,1,1)}[2, 2]$, or even surfaces in products of projective spaces. These constructions correspond to singular Weierstrass models in which the polynomials f, g in (4.2) are not as generic as possible, leading to the appearance of extra singularities. Consequently, already in $D = 8$, we can see that the resulting fibrations show different features compared to those of $\mathbb{P}_2^{(1,2,3)}[6]$. The $K3$ that they give rise to when fibered over \mathbb{P}_1 have less than 18 polynomial deformations, the missing moduli being provided by the Kähler classes of the extra A_1 singularities. This property leaves its track after fibration over another \mathbb{P}_1 .

4.1.1 Type B Models

We consider the fiber $\mathbb{P}_2^{(1,1,2)}[4]$ and derive the concrete structure of the fibrations as before. The ambient space is defined by

$$\begin{array}{cccccc} & z_1 & w_1 & z_2 & w_2 & x & y \\ \lambda & 1 & 1 & 0 & 0 & 2 & 4 \\ \mu & n & 0 & 1 & 1 & n+2 & 2n+4 \end{array} \quad (4.4)$$

and the hypersurface is given by the equation

$$y^2 = x^4 + f(z_1, w_1; z_2, w_2)x^2 + g(z_1, w_1; z_2, w_2)x + h(z_1, w_1; z_2, w_2) \quad (4.5)$$

We also note that, for $n \neq 0, 1$, these spaces coincide with $\mathbb{P}_4^{(1,1,n,n+2,2n+4)}[4n+8]$. These are precisely the last elements of the chains of type B models. The CY condition for this type of fibration changes, and forces n to be between 0 and 8. For this range of n , the fibrations have Hodge numbers matching our heterotic construction using $SU(2) \times U(1)$ backgrounds with instanton numbers distributed as $(9-n, 3; 9+n, 3)$. As we discussed in section 3, these models have an enhanced $U(1)$ gauge symmetry, confirming our expectations. Observe that the bound on n coincides in both constructions, being associated on the heterotic side to the collapse of some bundle structure due to lack of enough instantons to support it.

As mentioned earlier, the nature of the heterotic dual can be tested by studying the defining equations and the deformation of singularities near $z_1 = 0$ and $z_1 = \infty$. The fibration defined by (4.5) has an extra singularity responsible for the existence of the enhanced $U(1)$ symmetry. A detailed analysis of this type of singularity is still lacking. Nonetheless, we will see that the emergence of non-Abelian factors can be correctly deduced. Recognizing each particular singularity requires a change of variables to put eq. (4.5) into Weierstrass form. The starting point is the expansion of the polynomial coefficients in (4.5) in powers of z_1, w_1 , namely

$$\begin{aligned} f(z_1, w_1; z_2, w_2) &= \sum_{i=-2}^2 z_1^{2+i} w_1^{2-i} f_{4-ni}(z_2, w_2) \\ g(z_1, w_1; z_2, w_2) &= \sum_{j=-3}^3 z_1^{3+j} w_1^{3-j} g_{6-nj}(z_2, w_2) \\ h(z_1, w_1; z_2, w_2) &= \sum_{k=-4}^4 z_1^{4+k} w_1^{4-k} h_{8-nk}(z_2, w_2) \end{aligned} \quad (4.6)$$

A study of the generic singularities at $z_1 = \infty$ (there is no generic singularity at $z_1 = 0$) for each value of n reveals agreement with the pattern of heterotic terminal gauge groups as we now explain.

For $n = 0, 1, 2$ no singularity is found, as corresponds to having complete Higgsing of the gauge group in the heterotic side. The $n = 3$ case has an A_2 singularity, with no z_2 dependence, so that it maps to an $SU(3)$ gauge group without charged hypermultiplets. For $n = 4$ we get a D_4 singularity, leading to an $SO(8)$ gauge group without matter. The $n = 5$ case gives an E_6 singularity with quadratic dependence on z_2 so that in principle, monodromies around the singular fiber could break the symmetry to F_4 [29, 12]. However, it can be checked that the points at which the singularity worsens pair up to cancel the monodromy and the symmetry group E_6 remains unbroken with half hypermultiplets in the **27** representation. For $n = 6$ we find an E_6 singularity without z_2 dependence that maps to a heterotic E_6 gauge group with no matter. The singularity for $n = 7$ is of the E_7 type, and the linear z_2 dependence found is associated to a small instanton. Observe that this indicates the presence of the $U(1)$ background on the heterotic side since, in the absence of instantons, the gauge symmetry is $E_7(\times U(1))$ instead of E_8 . Finally, for $n = 8$ an E_7 symmetry is found, with no small instantons present. The same comments as in the precedent case apply here.

The counting of polynomial deformations near $z_1 = 0$ (*i.e.* polynomials with $i, j, k < 0$) and $z_1 = \infty$ ($i, j, k > 0$) can be performed as before. The number of singlets coming from each bundle is found to coincide with heterotic expectations. The result for complete Higgsing amounts to $[18(11+n)-133]$ and $[18(11-n)-133]$. This suggests that the instantons live in an E_7 subalgebra of E_8 , a fact that will be relevant for discussions in section 5.

The main conclusion from the above analysis is the perfect match between the F-theory construction and the pattern obtained in class B of heterotic compactification ($SU(2) \times U(1)$ bundles). The analysis for other fibers goes along the same lines. Below we will just sketch the main points.

4.1.2 Type C Models

Fibering $\mathbb{P}_2^{(1,1,1)}[3]$ over \mathbb{F}_n leads to the defining ambient space

$$\begin{array}{cccccc} & z_1 & w_1 & z_2 & w_2 & x & y \\ \lambda & 1 & 1 & 0 & 0 & 2 & 2 \\ \mu & n & 0 & 1 & 1 & n+2 & n+2 \end{array} \quad (4.7)$$

and the equation

$$\begin{aligned} x^3 + y^3 + f(z_1, w_1; z_2, w_2)xy + g(z_1, w_1; z_2, w_2)x + \\ g'(z_1, w_1; z_2, w_2)y + h(z_1, w_1; z_2, w_2) = 0 \end{aligned} \quad (4.8)$$

Again, for $n \neq 0, 1$, we find the hypersurfaces $\mathbb{P}_4^{(1,1,n,n+2,n+2)}[n+6]$ that give the last elements of the type C chains. The Hodge numbers match for all n the spectrum obtained from heterotic models with $SU(2) \times U(1)^2$ gauge backgrounds, as described in section 3. The bound on n due to the CY condition is $n \leq 6$ in this case, the same found in the heterotic construction.

A systematic analysis of the singularities for the different values of n leads to a pattern reproducing that of the gauge groups of heterotic type C compactifications. We find no singularity for $n = 0, 1, 2$. For $n = 3$ there is an A_2 singularity without z_2 dependence. For $n = 4$ there appears a D_4 singularity also without z_2 dependence. For $n = 5$ there is an E_6 gauge group and one small instanton due to the linear z_2 dependence of the singularity. Lastly, for $n = 6$ an E_6 singularity without z_2 dependence is found. A count of parameters perturbing the singularities gives also the number of neutral singlets in the model. As the conclusion is of some interest, we repeat the exercise explicitly. The polynomial deformations can be decomposed as follows

$$\begin{aligned} f(z_1, w_1; z_2, w_2) &= \sum_{i=-1}^1 z_1^{1+i} w_1^{1-i} f_{2-ni}(z_2, w_2) \\ g(z_1, w_1; z_2, w_2) &= \sum_{j=-2}^2 z_1^{2+j} w_1^{2-j} g_{4-nj}(z_2, w_2) \\ g'(z_1, w_1; z_2, w_2) &= \sum_{k=-2}^2 z_1^{2+k} w_1^{2-k} g'_{4-nk}(z_2, w_2) \\ h(z_1, w_1; z_2, w_2) &= \sum_{l=-3}^3 z_1^{3+l} w_1^{3-l} h_{6-nl}(z_2, w_2) \end{aligned} \quad (4.9)$$

One can check that there exist two extra singularities (so that we expect an enhanced $U(1)^2$ gauge symmetry), and that there are $[12(10+n) - 78]$ parameters deforming

the singularity at $z_1 = 0$ and $[12(10 - n) - 78]$ deforming that at $z_1 = \infty$ (in the case of complete Higgsing $n \leq 2$). This counting gives the right heterotic result, and suggests that the instantons in completely Higgsed models lie on an E_6 subalgebra of E_8 . Thus we find enough evidence to support the idea that these fibrations provide the duals of heterotic type C models.

4.1.3 Type D Models

We can repeat the construction with yet another fiber. The family of CY 3-folds obtained upon fibering $\mathbb{P}_3[2, 2]$ over \mathbb{F}_n , is described by

$$\begin{array}{ccccccc}
 & z_1 & w_1 & z_2 & w_2 & x & y & z \\
 \lambda & 1 & 1 & 0 & 0 & 2 & 2 & 2 \\
 \mu & n & 0 & 1 & 1 & n+2 & n+2 & n+2
 \end{array} \tag{4.10}$$

and the pair of equations

$$\begin{aligned}
 & x^2 + f(z_1, w_1; z_2, w_2)y + f'(z_1, w_1; z_2, w_2)z + \\
 & g(z_1, w_1; z_2, w_2)yz + h(z_1, w_1; z_2, w_2) = 0 \\
 & y^2 + z^2 + f''(z_1, w_1; z_2, w_2)x + h'(z_1, w_1; z_2, w_2) = 0
 \end{aligned} \tag{4.11}$$

After eliminating w_1 we obtain the CY spaces $\mathbb{P}_5^{(1,1,n,n+2,n+2,n+2)}[2n+4, 2n+4]$ for $n \neq 0, 1$. These coincide, for $n = 2, 4$, with $K3$ fibrations listed in [13] and the Hodge numbers coincide with the spectrum given by heterotic models with $SU(2) \times U(1)^3$ backgrounds (the D class of models) that display an enhanced $U(1)^3$ gauge group. A study of the structure of the CY similar to that performed for other fibers is also possible in this case. Again the results match those obtained from the heterotic type D constructions. We also notice that the count of neutral singlets in the case $n \leq 2$ gives $[8(9+n) - 45]$ and $[8(9-n) - 45]$, suggesting a $SO(10)$ structure for the instantons after Higgsing.

4.2 Conifold Transitions

We now address the question of the physical process connecting the different types of models A, B, C, D. The answer, of course, depends on the dimension of space-time, since, as noted in [5], in $D = 6$ these models should not be regarded as different. Since vector multiplets do not contain scalars, one cannot turn on vevs to change

the Kähler forms (unless they lie on the base \mathbb{F}_n), and the vector Coulomb phase is absent. Thus, the fibrations with fibers A, B, C, D are related by simply moving in the complex structure moduli space to different loci on which the Weierstrass models present extra singularities. However, in $D=4$ this is not the case, and a type IIA string compactified on such a singular space can smooth the singularity by simply turning on vevs for scalars associated to the Kähler structure of the CY, travelling to a new branch of the collective moduli space through such a conifold transition [31, 32]. We now turn to working out the details in a concrete example, showing that the CY spaces obtained with different fibers and fixed n are connected through this process. Also, we note that the transition can be mapped to an identical phenomenon in the dual heterotic picture.

As an example we consider the $n = 4$ ‘transversal’ chain formed with the fibrations of the different elliptic curves over \mathbb{F}_4 . This is

$$\begin{aligned} A : \mathbb{P}_4^{(1,1,4,12,18)}[36]_{271,7} &\rightarrow B : \mathbb{P}_4^{(1,1,4,6,12)}[24]_{164,8} \rightarrow \\ C : \mathbb{P}_4^{(1,1,4,6,6)}[18]_{111,9} &\rightarrow D : \mathbb{P}_5^{(1,1,4,6,6,6)}[12,12]_{76,10} \end{aligned} \quad (4.12)$$

where subscripts indicate the Hodge numbers. We start with the type IIA compactification with $(271,7)$ whose heterotic dual is obtained by embedding $SU(2)$ bundles with instanton numbers $(16,8)$ on $E_8 \times E_8$. The 4d gauge group is $SO(8) \times U(1)^4$ and it contains 272 singlets. To study the transition we choose a \mathbb{P}_2 inside the CY space, defined by $x_4 = x_5 = 0$. This submanifold will contain all the singular points at the conifold locus. The complex structure of the CY manifold can be adjusted so that only monomials containing at least one of the variables x_4, x_5 appear in the defining equation of the space that can then be written as

$$x_4 g_{24}(x_1, x_2, x_3, x_4, x_5) + x_5 h_{18}(x_1, x_2, x_3, x_4, x_5) = 0 \quad (4.13)$$

It then follows that this variety has singularities for the points such that $x_4 = x_5 = g_{24}(x_i) = h_{18}(x_i) = 0$. As noted before, all of them live on the selected \mathbb{P}_2 . The number of singular points is easily computed. Equations (4.13) have $24 \times 18 = 432$ solutions but only $108 = 432/4$ are distinct due to the scaling symmetries of the projective space. For generic choices of the polynomials g_{24}, h_{18} , the singularities have the typical conical structure with base $S^3 \times S^2$. The specific choice of complex structure has moved us to a conifold locus of dimension 164 (the number of independent polynomial deformations not smoothing the singularities)

and codimension 107 (the number of vanishing 3-cycles minus one homology relation among them).

Observe that the tuning of the complex structure is performed through vevs for the hypermultiplets. Hence, it maps to an un-Higgsing of the $U(1)$ gauge group on the heterotic side. It is illustrative to consider the spectrum of this heterotic model, listed in equation (3.17). The $U(1)$ gauge boson is associated to the Kähler class of the 2-cycle associated to the small resolution of the singularities. The 165 neutral singlets on the heterotic side correspond to the 164 type IIA fields (plus the dilaton) that can receive vevs without destroying the singularity (heterotic $U(1)$), while 108 fields can move us out of the singular locus (charged fields Higgsing the $U(1)$).

The singularities can be resolved by increasing the size of the S^2 's. Along this branch, a new Kähler class appears, and the 107 complex structure deformations associated to the 3-cycles are lost. This realizes the dual mechanism of the heterotic Higgsing with the scalar in the $U(1)$ vector multiplet, in which charged fields become massive. As the number of vector and hypermultiplets has changed, we have landed in the moduli space of another Calabi-Yau, in our case of Hodge numbers $(271 - 107, 7 + 1) = (164, 8)$. This spectrum and the precise mapping with the heterotic process suggest that we have arrived at the space $\mathbb{P}_4^{(1,1,4,6,12)}$ [24].

It is also possible to show that the complex structure of model B in (4.12) can be adjusted to develop conifold singularities whose resolution leads to the Hodge numbers of model C. The transition from C to D is similarly verified. The whole ‘transversal’ chain is also reproduced for other values of n ³ and is expected to be correct in general, even though some transitions may be harder to check. For example, in some cases b_{11} increases in several units, reflecting more homology relations among the 3-cycles. The interesting point in this discussion is to notice how all the connections in the vast web of models that we have explored can be studied from both the heterotic and the type IIA points of view, thus leading to a clearer picture of the way duality acts on these constructions.

³The $C \rightarrow D$ transition for $n = 2$ was noticed in ref. [33].

5 Phase Transitions

The $E_8 \times E_8$ compactifications obtained with $H \times U(1)^{8-d}$ backgrounds in $K3$ all have the same kind of strong coupling singularity found in the case $d = 8$. For example, for the B models with group $E_6 \times E_6 \times U(1)_D$, the anomaly polynomial (3.12) implies gauge kinetic terms of the form

$$-\frac{1}{3}(e^{-\phi} + \frac{n}{2}e^{\phi})\text{tr } F_1^2 - \frac{1}{3}(e^{-\phi} - \frac{n}{2}e^{\phi})\text{tr } F_2^2 - (e^{-\phi} + 3e^{\phi})f^2 \quad (5.1)$$

where we have set $k_1 = 9 + n$ and $k_2 = 9 - n$. Thus, the strong coupling singularity occurs again at the dilaton value (2.15). For models of type C and D we also find that the coupling of respectively the second $SO(10)$ and $SU(5)$ diverges at (2.15). We now discuss several features of the singularities following the analysis of refs. [22, 5, 34].

The singularity is signaled by the appearance of tensionless strings [23, 22, 24] that in the F-theory approach arise from a threebrane wrapping around a vanishing rational curve in the base [5]. Since we are dealing with elliptic fibrations over \mathbb{F}_n , general arguments [5] imply that when $n = 2$ there is actually no singularity since the collapsed \mathbb{F}_2 can be deformed into \mathbb{F}_0 and the gauge coupling is not singular when $n = 0$. This particular behaviour of the $n = 2$ case can be understood from the heterotic point of view as arising from the fact that, in that case, one can completely Higgs away the gauge group which is related to the singularity [21]. For $n = 1, 4$, general arguments also indicate that at the singularity there occurs a transition to a Higgs branch with no dilaton. We now review the supporting evidence of this type of transition for the new class of models discussed in the previous sections.

When $n = 1$, the transition is described by a change of base from \mathbb{F}_1 to \mathbb{P}_2 . Counting the change in parameters agrees with the results expected from a heterotic process in which the dilaton tensor multiplet disappears. For example, for models B, the fibration over \mathbb{F}_1 has an equation of the form (4.5) with coefficients f, g and h of bidegrees (4,6), (6,9) and (8,12) in λ and μ . The total number of monomials in (z_1, w_1, z_2, w_2) is 155 but there are 7 redundant parameters [34] due to transformations among the variables. Altogether there are 148 independent parameters. On the other hand, the fibration over \mathbb{P}_2 has equation

$$y^2 = x^4 + \tilde{f}(z_1, z_2, w_2)x^2 + \tilde{g}(z_1, z_2, w_2)x + \tilde{h}(z_1, z_2, w_2) \quad (5.2)$$

where now \tilde{f}, \tilde{g} and \tilde{h} are homogeneous functions of degrees 6, 9 and 12 in λ . There are 174 possible monomials in (z_1, z_2, w_2) and 9 redundant parameters so that the number of independent deformations is 165.

The above counting exercise basically amounts to a determination of the Hodge number b_{12} of the fibration over \mathbb{F}_1 and the fibration over \mathbb{P}_2 that happens to be $\mathbb{P}_4^{(1,1,1,3,6)}$ [12]. The difference $\Delta(b_{12}) = 165 - 148 = 17$ can be explained in the heterotic side by considering instanton numbers $(10, 3; 9, 3)$, and one less tensor multiplet, compared to the $n = 1$ instantons $(10, 3; 8, 3)$. The values of k_1 and k_2 are such that $E_6 \times E_6$ can be completely broken leaving behind the spectrum

$$\{54(\frac{1}{\sqrt{2}}) + \text{c.c.}\} + 166(0) \quad (5.3)$$

where we have taken into account the fact that when $n_T = 0$ there is one less neutral hypermultiplet. Notice that compared to (3.14) we do have 29 extra hypermultiplets as required by (2.17). However, twelve of them are charged and become massive in the Coulomb phase.

A similar analysis applies to models C in which the fibration of $\mathbb{P}_2[3]$ over \mathbb{P}_2 is $\mathbb{P}_4^{(1,1,1,3,3)}$ [9]. In this case we find $\Delta(b_{12}) = 112 - 101 = 11$, in agreement with the heterotic spectrum obtained for instanton numbers $(8, 3, 2; 7, 3, 2)$ and $n_T = 0$. For models D the fibration of $\mathbb{P}_3[2, 2]$ over \mathbb{P}_2 is the hypersurface $\mathbb{P}_5^{(1,1,1,3,3,3)}$ [6, 6] and we find $\Delta(b_{12}) = 77 - 70 = 7$. Again this is in agreement with results obtained in the heterotic side.

When n_T decreases by one, the number of 4d vector multiplets is lowered by one. Correspondingly, b_{11} decreases in one unit since there is one collapsing 2-cycle. Thus, in all models we can write

$$\begin{aligned} \Delta(b_{11}) &= -1 \\ \Delta(b_{12}) &= c_d - 1 \end{aligned} \quad (5.4)$$

where c_d is the Coxeter number of E_d and $d = 8, 7, 6$ and 5 for models A, B, C and D. The groups E_d do enter in the heterotic picture as follows. Notice that for $n = 1$, complete Higgsing of the non-Abelian groups is possible in all models and this can be achieved by instantons of $E_d \times U(1)^{8-d}$ that leave $U(1)^{8-d}$ unbroken in each E_8 (before further breaking to the diagonal combinations). In fact, the transition to $n_T = 0$ occurs when $k_2 \rightarrow k_2 + 1$, where k_2 corresponds to an E_d instanton. In the F-theory picture, the E_d groups appear because when the 2-cycle collapses in

$\mathbb{F}_1 \rightarrow \mathbb{P}_2$, there also shrinks a 4-cycle of del Pezzo type [5]. In turn, this del Pezzo surface is related to the form of the singularity at $w_1 = 0$ [5]. For example, for models C with $d = 6$, from (4.8) we see that, setting say $z_1 = 1$, the singularity is locally a hypersurface in \mathbb{C}^4 with leading cubic terms. Similarly, for models D with $d = 5$, (4.11) implies that the singularity is locally the intersection of two quadratic equations in \mathbb{C}^5 .

It is also possible to probe the current algebra carried by the tensionless string that develops when an instanton shrinks [23] in the reverse transition $\mathbb{P}_2 \rightarrow \mathbb{F}_1$. In F-theory a rank one current algebra is supported at the intersection of a type IIB 3-brane and a type IIB 7-brane [34]. The idea is then to determine the number of 7-branes that meet the 2-cycle blanketed by the 3-brane. In turn this can be done by counting the parameters of the fibration restricted to $w_1 = 0$ [4, 34]. For example, for models B, from eq. (4.6) we find 3 parameters in f , 4 in g and 5 in h . Eliminating the redundancies due to linear transformations of (z_2, w_2) leaves 8 independent parameters. This indicates then that the 3-brane intersects eight 7-branes so that the current algebra has rank eight. The same result readily follows for models C and D.

Existence of a Higgs branch with zero tensor multiplets is also expected in the strong coupling transition for the $n = 4$ case, on the basis of anomaly cancellation arguments [22] and F-theory computations [5]. Since, in the latter approach, the transition corresponds to a deformation of the base of the fibration from \mathbb{F}_4 to \mathbb{P}_2 , it follows that such kind of transitions will be possible not only for A, but also for B, C and D models. Actually, it is evident that the change in the Hodge numbers for the associated CY spaces follows the rule

$$\begin{aligned}\Delta(b_{11}) &= -4 \\ \Delta(b_{12}) &= 1\end{aligned}\tag{5.5}$$

which can be understood as follows. As the tensor multiplet disappears, anomaly cancellation conditions force the appearance of 29 new hypermultiplets, 28 of them are employed in Higgsing the $SO(8)$ gauge symmetry and one remains in the final spectrum providing for the increase in b_{21} . In this process 4 Cartan generators are lost, thus explaining the change in b_{11} . It is important to notice the relevance of the $SO(8)$ symmetry for this counting to work (on the F-theory side, the existence of the corresponding D_4 singularity is discussed in [5]). As remarked in previous

sections, this requirement is fulfilled by all $n = 4$ models. An interesting point in this discussion is that the new hypermultiplets appearing in the transition seem to be charged under the terminal gauge group $SO(8)$. Also, in the new branch there is no generic gauge symmetry as corresponds to the F-theory fibration over \mathbb{P}_2 . The mechanism of smoothing the singularity gives a hint about how this occurs. As described in [5], it is related to the Z_2 quotient of the deformation of $\mathbb{P}_2 \rightarrow \mathbb{P}_1 \times \mathbb{P}_1$. Since in this process the instanton numbers embedded in each E_8 change from a $(14, 10)$ to a $(12, 12)$ distribution, we expect a similar change in the initial $(16, 8)$ instanton distribution for $n = 4$. In this way the bundle in the second E_8 ends up with enough instantons to achieve complete Higgsing of the gauge group.

A comment concerning transitions to $D = 6$ models with no tensor multiplets is in order. Just looking at the spectrum, from a purely 4d point of view, it may be difficult to disentangle whether the dual of a certain type II CY compactification is a perturbative ($n_T = 1$) or a non-perturbative ($n_T = 0$) heterotic vacuum. Let us consider the $n = 4$ case. For the A chain this can be obtained by embedding instanton numbers $(k_1, k_2) = (16, 8)$ in $E_8 \times E_8$. Higgsing as much as possible the second E_8 we arrive at a $D = 6$ model with gauge group $E_7 \times SO(8)$ and hypermultiplets transforming as $6(\mathbf{56}, \mathbf{1}) + 69(\mathbf{1}, \mathbf{1})$. In addition there is a $D = 6$ tensor multiplet containing the dilaton. Now consider the model obtained by embedding instantons with $(k_1, k_2) = (16, 9)$ in $E_8 \times E_8$. This is the final stage of a model in which the original tensor multiplet has been absorbed at the M-theory boundary and has been converted into an instanton in the second E_8 . Thus this model is continuously connected to the previous one. Since there is no dilaton to make a perturbative expansion this is a non-perturbative vacuum. We can Higgs in steps the second E_7 of this theory. If we stop at an $SO(10)$ stage, the gauge group will be $E_7 \times SO(10)$ and it is easy to check that there will be hypermultiplets transforming as $6(\mathbf{56}, \mathbf{1}) + 69(\mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{16}) + 3(\mathbf{1}, \mathbf{10})$. Now, the point is that if we further compactify these two models on T^2 , Higgs completely the first E_7 and go to the Coulomb phase, we arrive in both cases to a $N = 2$ model with the same number of vector multiplets and hypermultiplets, corresponding to a type II compactification on a CY with $(b_{21}, b_{11}) = (271, 7)$. For the perturbative heterotic vacuum the seven vector multiplets correspond to $7 = S + T + U + \text{rank}(SO(8))$ whereas for the non-perturbative model one has $7 = T + U + \text{rank}(SO(10))$, in an obvious notation. We know that these two models are connected through a transition $n_T = 1 \rightarrow n_T = 0$.

Hence, one can argue that the $SO(8)$ group of the first model can combine with the dilaton vector multiplet to get a non-perturbatively enhanced $SO(10)$.

6 Final Comments and Conclusions

In the heterotic constructions of section 3, Abelian backgrounds played an essential role. They provide a systematics for deriving chains of different types, each type corresponding to the inclusion of a given number of $U(1)$'s. For a given set of instanton numbers, specified by n , many Higgsing branches can be followed. Continuous flow from an n fixed branch of a given type to another type is achieved by Higgsing $U(1)$ gauge groups. These processes have a dual description in terms of transitions in the space of CY spaces. It must be emphasized that the full web of dual theories is quite intricate. Identifying precisely special points such as the terminal A,B,C and D models provides a useful handle in exploring this web.

We have shown that the process of changing the fiber in F-theory compactifications is associated to the appearance of enhanced gauge symmetries arising from $E_8 \times E_8$. As one can embed a larger number of $U(1)$ backgrounds in E_8 on the heterotic side, we expect to find further families of CY spaces associated to other F-theory fibers. Also, once the last elements of the chains have been understood, all models corresponding to un-Higgsing of gauge symmetries should be derivable using the techniques presented in refs. [10, 12], leading to an extended web of models on the F-theory side. It would be interesting, for instance, to study type B and C duals with enhanced $SU(r)$ groups and compare their Hodge numbers with those implied by the heterotic analysis of section 3.

The heterotic models discussed to large extent, all arise from $E_8 \times E_8$ compactifications. However, in some situations, there appear suggestive correlations when $SO(32)$ compactifications are examined. For instance, it is well known that starting with the standard embedding in $SO(32)$ leads, for generic moduli, to a model with Hodge numbers $(271, 7)$ and a matter-free terminal $SO(8)$, the same result found in the terminal $n = 4$ A model. In fact, the equivalence between both constructions was established in [28] by using T-duality arguments. Moreover, the same authors show that the symmetric instanton embedding $(12, 12)$, *i.e.* the $n = 0$ type A case is equivalent to an $SO(32)$ compactification without vector structure. This corresponds to the Type I string model elaborated in [35]. We have found extra examples

that suggest additional relations between $SO(32)$ and $E_8 \times E_8$ compactifications.

The first example is a six-dimensional Z_3 orbifold compactification accompanied by the embedding of the shift $V = \frac{1}{3}(-2, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 0, 0, 0, 0, 0)$ in the $Spin(32)/Z_2$ lattice. The resulting model has gauge group $SU(11) \times SO(10) \times U(1)$ and massless hypermultiplet spectrum given by

$$\begin{aligned}\theta^0 : & \quad (\mathbf{11}, \mathbf{10}, -1) + (\mathbf{55}, \mathbf{1}, -2) + 2(\mathbf{1}, \mathbf{1}, 0) \\ \theta^1 : & \quad 9[(\mathbf{11}, \mathbf{1}, \frac{2}{3}) + (\mathbf{1}, \mathbf{1}, \frac{5}{3}) + (\bar{\mathbf{10}}, \mathbf{1}, -\frac{4}{3})]\end{aligned}\tag{6.1}$$

The gauge group can be completely Higgsed away, leading to $(243, 3)$ Hodge numbers. Moreover, first breaking $SO(10)$ fully and then performing a cascade breaking of $SU(11)$, the chain

$$\dots \rightarrow (193, 8) \rightarrow (204, 6) \rightarrow (215, 5) \rightarrow (226, 4) \rightarrow (243, 3)\tag{6.2}$$

is obtained. This is the same set of numbers found for $n = 1$ type A models with instanton numbers $(13, 11)$, if the first E_8 (with $k_1 = 13$) is completely Higgsed and the second is broken sequentially. Interestingly enough, an alternative Higgs branch can be followed through level two models with adjoints, due to the presence of the $(\mathbf{11}, \mathbf{10})$ representation. The Hodge numbers $(70, 6) \rightarrow (101, 5) \rightarrow (148, 4) \rightarrow (243, 3)$ are derived in this way. This corresponds to transversal transitions $D \rightarrow C \rightarrow B \rightarrow A$ among the $n = 1$ terminal elements.

Constructions in terms of semisimple bundles in $SO(32)$ are also interesting. For instance, by embedding an $SU(8)_{(k_1)} \times SU(8)_{(k_2)}$ bundle in $SO(32)$ it is easy to see that for instanton numbers $(k_1, k_2) = (12, 12), (13, 11), (14, 10)$, full Higgsing is possible ending in the $(243, 3)$ model. Another interesting example starts with background $SU(2)_{(4)} \times SU(2)_{(6)} \times SU(2)_{(14)}$ in $SO(32)$ to obtain observable group $SU(2)^3 \times SO(20)$. Higgsing through steps similar to those discussed in Chapter 3, the bundle may be deformed to $SU(2)_{(4)} \times SU(3)_{(6)} \times SO(14)_{(14)} \rightarrow SU(3)_{(6)} \times SO(18)_{(18)} \rightarrow SO(24)_{(24)}$. Using eq. (3.29) to compute the number of moduli we encounter the sequence $(111, 9) \rightarrow (164, 8) \rightarrow (271, 7)$, corresponding to transitions through last elements of different types for $n = 4$.

In conclusion, in this paper we have studied new branches of $D = 6, 4$ heterotic string compactifications obtained by including Abelian backgrounds on the $E_8 \times E_8$ heterotic string. The corresponding type II duals can be derived from F-theory by changing appropriately the elliptic fiber. Our procedure allows us to explicitly

construct the heterotic duals of many type IIA compactifications on $K3$ fibrations whose duals were previously unknown. It also allows us to understand the existence of some chains of models which were conjectured to be connected in [7]. The connections between the different types of chains of models are understood in terms of the Higgsing of $U(1)$'s in the heterotic side and conifold transitions from the type II side. We also identify new $D = 6$ models in which transitions from theories with one tensor multiplet to zero tensor multiplets occur. Other interesting features appear in our class of models. In particular, there are anomalous $U(1)$'s that are in fact Higgsed away by swallowing zero modes of the antisymmetric B_{MN} field. A similar phenomenon was recently reported in [28].

Although most of the work reported here is related to compactifications of the $E_8 \times E_8$ heterotic string, it is clear that related models may be obtained from $SO(32)$. It would be interesting to study also these models and their connections to type II compactifications.

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n	A		B		C	
	(b_{12}, b_{11})	Weights	(b_{12}, b_{11})	Weights	(b_{12}, b_{11})	Weights
2	(138,6)	(1,1,2,6,8,10)				
	(161,5)	(1,1,2,6,8)	(102,6)	(1,1,2,4,6,6)		
	(190,4)	(1,1,2,6,10)	(121,5)	(1,1,2,4,6)	(82,6)	(1,1,2,4,4,6)
	(243,3)	(1,1,2,8,12)	(148,4)	(1,1,2,4,8)	(101,5)	(1,1,2,4,4)
3	(124,8)	(1,1,3,7,9,11)				
	(151,7)	(1,1,3,7,9)	(96,8)	(1,1,3,5,7,9)		
	(186,6)	(1,1,3,7,12)	(119,7)	(1,1,3,5,7)	(80,8)	(1,1,3,5,5,7)
	(251,5)	(1,1,3,10,15)	(152,6)	(1,1,3,5,10)	(103,7)	(1,1,3,5,5)
4	(122,10)	(1,1,4,8,10,12)				
	(153,9)	(1,1,4,8,10)	(98,10)	(1,1,4,6,8,10)		
	(194,8)	(1,1,4,8,14)	(125,9)	(1,1,4,6,8)	(84,10)	(1,1,4,6,6,8)
	(271,7)	(1,1,4,12,18)	(164,8)	(1,1,4,6,12)	(111,9)	(1,1,4,6,6)
5	(124,10)	(1,1,5,9,11,13)				
	(159,9)	(1,1,5,9,11)	(102,12)	(1,1,5,7,9,11)		
	(206,8)	(1,1,5,9,16)	(133,11)	(1,1,5,7,9)	(89,13)	(1,1,5,7,7,9)
	(295,7)	(1,1,5,14,21)	(178,10)	(1,1,5,7,14)	(120,12)	(1,1,5,7,7)
6	(128,12)	(1,1,6,10,14)				
	(167,11)	(1,1,6,10,12)	(108,12)	(1,1,6,8,10,12)		
	(220,10)	(1,1,6,10,18)	(143,11)	(1,1,6,8,10)	(96,12)	(1,1,6,8,8,10)
	(321,9)	(1,1,6,16,24)	(194,10)	(1,1,6,8,16)	(131,11)	(1,1,6,8,8)
7	(133,13)	(1,1,7,11,13,15)				
	(176,12)	(1,1,7,11,13)	(114,14)	(1,1,7,9,11,13)		
	(235,11)	(1,1,7,11,20)	(153,13)	(1,1,7,9,11)		
	(348,10)	(1,1,7,18,27)	(210,12)	(1,1,7,9,18)		
8	(139,13)	(1,1,8,12,14,16)				
	(186,12)	(1,1,8,12,14)	(121,13)	(1,1,8,10,12,14)		
	(251,11)	(1,1,8,12,22)	(164,12)	(1,1,8,10,12)		
	(376,10)	(1,1,8,20,30)	(227,11)	(1,1,8,10,20)		
9	(145,17)	(1,1,9,13,15,17)				
	(196,16)	(1,1,9,13,15)				
	(267,15)	(1,1,9,13,24)				
	(404,14)	(1,1,9,22,33)				
10	(152,16)	(1,1,10,14,16,18)				
	(207,15)	(1,1,10,14,16)				
	(284,14)	(1,1,10,14,26)				
	(433,13)	(1,1,10,24,36)				
11	(159,15)	(1,1,11,15,17,19)				
	(218,14)	(1,1,11,15,17)				
	(301,13)	(1,1,11,15,28)				
	(462,12)	(1,1,11,26,39)				
12	(166,14)	(1,1,12,16,18,20)				
	(229,13)	(1,1,12,16,18)				
	(318,12)	(1,1,12,16,30)				
	(491,11)	(1,1,12,28,42)				

Table 3: Hodge numbers for the chains of CY spaces.